Übungsstunde 11:

Themen:

- Definitheit von Matrizen
- Singulärwertzerlegung (SVD) für die Ausgleichsrechnung

Definitheit: Hurwitz-Kriterium / Sylvester-Kriterium

$\underline{\underline{A}}$ symmetrisch, quadratisch ist positiv definit, falls alle führenden Hauptminoren von $\underline{\underline{A}}$ positiv sind.

$$\underline{x}^T \underline{\underline{A}}\,\underline{x} > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\underbrace{\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1i} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{i1} & & & a_{ii} \end{bmatrix}}_{\text{Hauptminoren}} > 0 \quad \forall i \in [1,n], \quad \underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

Bsp. 11.1:

$$\underline{\underline{A}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 8 & -4 \\ 0 & -4 & 3 \end{bmatrix}$$

$$a_1 = \det[2] = 2 > 0$$

$$a_2 = \det \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} = 16 > 0$$

$$a_3 = \det \begin{bmatrix} \boxed{2^+} & 0^- & 0^+ \\ 0^- & 8^+ & -4^- \\ 0^+ & -4^- & 3^+ \end{bmatrix} = 2 \cdot \det \begin{bmatrix} 8 & -4 \\ -4 & 3 \end{bmatrix} = 2 \cdot (24 - 16) = 16 > 0$$

$\leadsto \underline{\underline{A}}$ pos. definit nach Hurwitz

Vorzeichen $<0, >0, <0, >0$

$\rightarrow \underline{\underline{A}}$ neg. definit

Singulärwertzerlegung (SVD) $\quad \underline{\underline{A}}\,\underline{x} = \underline{c}$

$$\underline{\underline{A}} = \underline{\underline{U}} \cdot \underline{\underline{S}} \cdot \underline{\underline{V}}^T, \quad \underline{\underline{S}} = \begin{bmatrix} \hat{\underline{\underline{S}}} \\ 0 \end{bmatrix}, \quad \underline{\underline{U}} \,\&\, \underline{\underline{V}} \text{ orthogonal}$$

Bei der Ausgleichsrechnung möchten wir $\min\limits_{\underline{x} \in \mathbb{R}^n} \|\underline{r}\|_2^2$

$$\|\underline{r}\|_2^2 = \|\underline{\underline{A}}\,\underline{x} - \underline{c}\|_2^2 = \|\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{c}\|_2^2$$

$$= \|\underline{\underline{U}}(\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{\underline{U}}^T \underline{c})\|_2^2 \qquad \|\underline{\underline{U}}\|_2^2 = 1, \text{ da } U \text{ orth.}$$

$$= \|\underline{\underline{U}}\|_2^2 \, \|\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{\underline{U}}^T \underline{c}\|_2^2$$

$$= \|\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underbrace{\underline{\underline{U}}^T \underline{c}}\|_2^2$$

$$= \|\underbrace{\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} - \underline{d}}_{\begin{bmatrix} \hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} \\ 0 \end{bmatrix} - \begin{bmatrix} \underline{d}_0 \\ \underline{d}_1 \end{bmatrix}}\|_2^2 = \underbrace{\|\hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} - \underline{d}_0\|_2^2}_{\text{Kann gelöst werden!}} + \underbrace{\|\underline{d}_1\|_2^2}_{\text{Der Fehler / das Residuum!}}$$

$$\hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} = \underline{d}_0$$

2 Möglichkeiten / Fälle:

$\Rightarrow \quad \hat{\underline{\underline{S}}}$ invertierbar: $\quad \underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{-1} \underline{d}_0$

$\quad\;\; \hat{\underline{\underline{S}}}$ nicht-inv.: $\quad \underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{+} \underline{d}_0 \qquad \hat{\underline{\underline{S}}}^{+}$ ist die Pseudo-Inverse von $\hat{\underline{\underline{S}}}$!

$\underline{\underline{U}}$: EV von $\underline{\underline{A}} \cdot \underline{\underline{A}}^T$ $\qquad$ Der Grösse nach sortiert absteigend

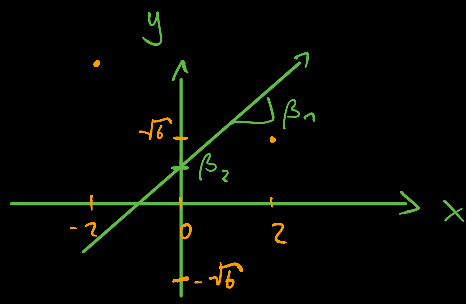
$\underline{\underline{V}}$: EV von $\underline{\underline{A}}^T \cdot \underline{\underline{A}}$

$\underline{\underline{S}}$: $\operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_p})$, λ_i EW von $\underline{\underline{A}}\,\underline{\underline{A}}^T$ o. $\underline{\underline{A}}^T\underline{\underline{A}}$, $p = \min\{m, n\}$

$$\underline{v}^{(i)} = \frac{\underline{\underline{A}}^T \underline{u}^{(i)}}{\sigma^{(i)}}, \quad \underline{u}^{(i)} = \frac{\underline{\underline{A}}\,\underline{v}^{(i)}}{\sigma^{(i)}}, \quad \sigma^{(i)} = \sqrt{\lambda_i}$$

Bsp.: Prüfung HS 16

$$y = \beta_1 x + \beta_2 \quad \Longleftrightarrow$$



x_i	2	0	-2
y_i	$\sqrt{6}$	$-\sqrt{6}$	$2\sqrt{6}$

$$\sum_{i=1}^{3} |y_i - (\beta_1 x_i + \beta_2)|^2$$ soll minimiert werden

$$\underline{\underline{A}}\,\underline{\beta} = \underline{c}$$

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix}, \quad \underline{\underline{A}} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T$$

$$\underline{\underline{U}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -3 \\ 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix}$$

$$\text{EW: } \det(\underline{\underline{A}}\,\underline{\underline{A}}^T - \lambda \underline{\underline{I}}) = \det \begin{bmatrix} 5-\lambda & 1 & -3 \\ 1 & 1-\lambda & 1 \\ -3 & 1 & 5-\lambda \end{bmatrix}$$

$$= (5-\lambda)\det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 5-\lambda \end{bmatrix} - 1\det\begin{bmatrix} 1 & -3 \\ 1 & 5-\lambda \end{bmatrix} - 3\det\begin{bmatrix} 1 & -3 \\ 1-\lambda & 1 \end{bmatrix}$$

$$= (5-\lambda)\left[(1-\lambda)(5-\lambda) - 1\right] - (5-\lambda+3) - 3(1-(3\lambda-3))$$

$$= (1-\lambda)(5-\lambda)^2 - (5-\lambda) + \lambda - 8 - 12 + 9\lambda$$

$$= (1-\lambda)(25 - 10\lambda + \lambda^2) + 11\lambda - 25$$

$$= -\lambda^3 + 11\lambda^2 - 24\lambda \overset{!}{=} 0 \qquad \lambda_3 = 0$$

$$= -\lambda(\lambda^2 - 11\lambda + 24) = 0$$

$$= -\lambda(\lambda-3)(\lambda-8) = 0 \qquad \lambda_2 = 3 \qquad \lambda_1 = 8$$

$$\left.\begin{array}{l} \Rightarrow \lambda_1 = 8,\ \lambda_2 = 3,\ \lambda_3 = 0 \\ \Rightarrow \sigma_1 = \sqrt{\lambda_1} = \underline{2\sqrt{2}},\ \sigma_2 = \underline{\sqrt{3}} \end{array}\right\} \quad \underline{\underline{S}} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

$$\hat{\underline{\underline{S}}}$$

EV: $(\underline{\underline{A}}\underline{\underline{A}}^T - \lambda \underline{\underline{I}})x = 0$

$\lambda_1 = 8$:

$$\left[\begin{array}{ccc|c} -3 & 1 & -3 & 0 \\ 1 & -7 & 1 & 0 \\ -3 & 1 & -3 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -7 & 1 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_3 = t \in \mathbb{R} \\ x_2 = 0 \\ x_1 = -x_3 = -t \end{array}$$

$$\Rightarrow E_8 = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} = \underline{\text{span}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}}$$

$\lambda_2 = 3$:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 \\ -3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_3 = s \in \mathbb{R} \\ x_2 = s \in \mathbb{R} \\ x_1 = s \in \mathbb{R} \end{array}$$

$$\Rightarrow E_3 = \text{span}\underline{\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$$

$\lambda_3 = 0$:

$$\left[\begin{array}{ccc|c} 5 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 1 & 5 & 0 \end{array}\right] \xrightarrow{G.} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_3 = u \in \mathbb{R} \\ x_2 = -2u \\ x_1 = u \end{array}$$

$$\Rightarrow E_0 = \text{span}\underline{\left\{\frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}}$$

$$\Rightarrow \underline{\underline{U}} = \underline{\frac{1}{\sqrt{6}}\begin{bmatrix} -\sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}}$$

$$\underline{v}^{(1)} = \frac{\underline{\underline{A}}^T \underline{u}^{(1)}}{\sigma^{(1)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{2\sqrt{2}} = \frac{1}{4} \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}}$$

$$\underline{v}^{(2)} = \frac{\underline{\underline{A}}^T \underline{u}^{(2)}}{\sigma^{(2)}} = \frac{\begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\sqrt{3}} = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$\Rightarrow \underline{\underline{V}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{c}$$

$$\underline{\underline{S}}\,\underline{\underline{V}}^T \underline{x} = \underline{\underline{U}}^T \underline{c} = \underline{d} = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$$

$$\Rightarrow \hat{\underline{\underline{S}}}\,\underline{\underline{V}}^T \underline{x} = \underline{d_0}$$

$$\underline{d} = \underline{\underline{U}}^T \underline{c} = \frac{1}{\sqrt{6}} \begin{bmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ * & * & * \end{bmatrix} \begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \\ 2\sqrt{6} \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 3\sqrt{2} \\ 4\sqrt{3} \\ * \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \\ * \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \underline{d_0} \\ 5 \ \} d_1 \to \text{Res.} \end{matrix}$$

$$\Rightarrow \underline{x} = \underline{\underline{V}}\,\hat{\underline{\underline{S}}}^{-1} \underline{d_0} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 2\sqrt{2} \end{bmatrix} = \underline{\underline{\begin{bmatrix} -\frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{2\sqrt{2}}{\sqrt{3}} \end{bmatrix}}} \begin{matrix} \beta_1 \\ \beta_2 \end{matrix}$$

$$\hat{\underline{\underline{S}}}^{-1} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}^{-1} = \frac{1}{\det(\hat{S})} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} = \frac{1}{2\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Allgemein: $$\hat{\underline{\underline{S}}}^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & & \emptyset \\ & \frac{1}{\sigma_2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_n} \end{bmatrix}$$

$$\hat{\underline{\underline{S}}}^{+} = \begin{bmatrix} \frac{1}{\sigma_1} & & & & \emptyset \\ & \frac{1}{\sigma_2} & 0 & & \\ \emptyset & & 0 & \ldots & 0 \end{bmatrix}$$

Zusammenhang zwischen U & V:

$$\underline{\underline{A}}^T \underline{\underline{A}}\, \underline{v}^{(i)} = \lambda\, \underline{v}^{(i)}$$

Richtung stimmt

$$(\underline{\underline{A}}\,\underline{\underline{A}}^T)\underbrace{\underline{\underline{A}}\,\underline{v}^{(i)}}_{\underline{\tilde{u}}^{(i)}} = \lambda \underbrace{\underline{\underline{A}}\,\underline{v}^{(i)}}_{\underline{\tilde{u}}^{(i)}} = \lambda\, \tilde{u}^{(i)} \quad \Rightarrow \quad \underline{u}^{(i)} = \frac{\underline{\tilde{u}}^{(i)}}{\sigma^{(i)}} = \frac{\underline{\underline{A}}\,\underline{v}^{(i)}}{\sigma^{(i)}}$$

Normieren, damit $\underline{\underline{U}}/\underline{\underline{V}}$ orthogonal wird

$$\underline{\underline{A}}\,\underline{\underline{A}}^T \underline{u}^{(i)} = \lambda\, \underline{u}^{(i)}$$

$$(\underline{\underline{A}}^T \underline{\underline{A}})\underbrace{\underline{\underline{A}}^T \underline{u}^{(i)}}_{\underline{\tilde{v}}^{(i)}} = \lambda \underbrace{\boxed{\underline{\underline{A}}^T \underline{u}^{(i)}}}_{\underline{\tilde{v}}^{(i)}} = \lambda\, \underline{\tilde{v}}^{(i)} \quad \Rightarrow \quad \underline{v}^{(i)} = \frac{\underline{\tilde{v}}^{(i)}}{\boxed{\sigma^{(i)}}}$$

Mit welchem Faktor muss ich normieren?

Deswegen wird mit den Singulärwerten normiert!

$$\|\underline{\underline{A}}^T \underline{u}\|_2 = \|(\underline{\underline{U}}\,\underline{\underline{S}}\,\underline{\underline{V}}^T)^T \underline{u}\|_2 = \|\underline{\underline{V}}\,\underline{\underline{S}}^T \underline{\underline{U}}^T \underline{u}\|_2$$

U & V orthogonal → verändern Länge nicht!

$$= \|\underline{\underline{S}}^T \underline{u}\|_2$$

S diagonal!

$$= \|\sigma^{(i)} \underline{u}^{(i)}\|_2 = \boxed{\|\sigma^{(i)}\|_2}$$

u hat Länge 1!

↳ Ergebnis hat also Länge der Singulärwerte!

$$\|\underline{\underline{A}}\,\underline{v}\|_2 = \ldots$$